

The Order Completion Method for Systems of Nonlinear PDEs Revisited

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Abstract. In this paper we presents further developments regarding the enrichment of the basic Theory of Order Completion as presented in (Oberguggenberger and Rosinger, 1994). In particular, spaces of generalized functions are constructed that contain generalized solutions to a large class of systems of continuous, nonlinear PDEs. In terms of the existence and uniqueness results previously obtained for such systems of equations (van der Walt, 2008 [2]), one may interpret the existence of generalized solutions presented here as a regularity result.

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1. Introduction

Consider a possibly nonlinear PDE, of order at most m , of the form

$$T(x, D)u(x) = f(x), x \in \Omega \subseteq \mathbb{R}^n \quad (1)$$

with the righthand term f a continuous function of $x \in \Omega$, and the partial differential operator $T(x, D)$ defined by some jointly continuous mapping

$$F : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}$$

through

$$T(x, D)u(x) = F(x, u(x), \dots, D^\alpha u(x), \dots), |\alpha| \leq m \quad (2)$$

It is well know that an equation of the form (1) through (2) may, in general, fail to have a classical solution $u \in \mathcal{C}^m(\Omega)$. Moreover, there is in fact a *physical interest* in solutions to (1) that are not classical. From there the interest in generalized solutions to nonlinear PDEs.

A long established idea in analysis is to obtain the existence of generalized solutions to (1) by associating with the partial differential operator $T(x, D)$ a mapping

$$T : X \ni u \mapsto Tu \in Y \quad (3)$$

where X is a relatively small space of classical functions on Ω , and Y is some suitable space of functions with $f \in Y$. Appropriate topological structures, typically a norm or locally convex topology, are defined on X and Y so that the mapping T is uniformly continuous with respect to these customary structures. Generalized solutions to (1) are obtained by constructing the completions X^\sharp and Y^\sharp of X and Y , respectively, and extending the mapping T to a mapping

$$T^\sharp : X^\sharp \rightarrow Y^\sharp \quad (4)$$

A solution of the equation

$$T^\sharp u^\sharp = f \quad (5)$$

where the unknown u^\sharp ranges over X^\sharp , is considered a generalized solution of (1).



As mentioned, the customary structures on the spaces X and Y in (3) are typically locally convex linear space topologies, or even normable topologies. However, such methods, involving the customary linear topological spaces of generalized functions, appear ineffective in providing a *general* and *type independent* theory for the existence and regularity of solutions to nonlinear PDEs. This apparent failure of the usual methods of linear functional analysis in the study of nonlinear PDEs is ascribed to the ‘complicated geometry of \mathbb{R}^n ’ (Arnold, 2004). Moreover, in view of the above mentioned inability of linear functional analysis and other customary methods to yield such a general approach, it is widely held that it is in fact *impossible*, or at the very best *highly unlikely*, that such a theory exists.

This, as will be seen in the sequel, is in fact a misunderstanding. In this regard, we should mention that there are currently two *general* and *type independent* theories for the existence and regularity of generalized solutions of nonlinear PDEs. The Central Theory of PDEs (Neuberger, 1997) through (Neuberger, 2005) is based on a generalized method of steepest descent in suitably constructed Hilbert spaces. This method is fully type independent, that is, the particular form of the operator that defines the equation is not used, and rather general, but as of yet it is not universally applicable. In those cases where the method has been applied, it has resulted in impressive numerical results. The Order Completion Method (Oberuggenberger and Rosinger, 1994) and (Anguelov and Rosinger, 2005), on the other hand, constructs generalized solutions to a large class of nonlinear PDEs in the Dedekind order completion of suitable spaces of functions. The essential feature of both methods is that the spaces of generalized functions are tied to the particular nonlinear partial differential operator $T(x, D)$. Moreover, the underlying ideas upon which they are based apply to situations that are far more general than PDEs, this being exactly the reason for their respective type independent power.

Recently, (van der Walt, 2008 [2]) the Order Completion Method was recast in the setting of uniform convergence spaces (Beattie and Butzmann, 2002). For a system of K nonlinear PDEs, each of order at most m , in K unknown functions of the form

$$\mathbf{T}(x, D) \mathbf{u}(x) = \mathbf{f}(x), \quad x \in \Omega \subseteq \mathbb{R}^n, \quad (6)$$

where Ω is open, and \mathbf{f} is a continuous, K -dimensional vector valued function on Ω with components $f_1, \dots, f_K : \Omega \rightarrow \mathbb{R}$, generalized solutions are constructed as the elements of the completion of a suitable uniform convergence space. In particular, subject to a mild assumption on the PDE (6), namely

$$\begin{aligned} \forall x \in \Omega : \\ \mathbf{f}(x) \in \text{int}\{\mathbf{F}(x, \xi) : \xi \in \mathbb{R}^M\} \end{aligned} \quad (7)$$

where $\mathbf{F} : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^K$ is the jointly continuous function that defines the system of PDEs (6) through

$$\mathbf{T}(x, D) \mathbf{u}(x) = \mathbf{F}(x, u_1(x), \dots, u_K(x), \dots, D^\alpha u_i(x), \dots), \quad |\alpha| \leq m \text{ and } i = 1, \dots, K, \quad (8)$$

we obtain the existence and uniqueness of generalized solutions to (6). Moreover, the generalized solution satisfies a blanket regularity as it may be assimilated with nearly finite normal lower semi-continuous functions. In particular, there is an uniformly continuous embedding from the space of generalized solutions into the space of nearly finite normal lower semi-continuous functions.

It should be noted that the assumption (7) is hardly a restriction on the class of PDEs to which the method applies. Indeed, every linear PDEs, as well as most nonlinear PDEs of applicative interest satisfy it trivially since, in these cases,

$$\{\mathbf{F}(x, \xi) : \xi \in \mathbb{R}^M\} = \mathbb{R}^K.$$

Therefore, the Order Completion Method (Oberguggenberger and Rosinger, 1994) and the pseudo-topological version of the theory (van der Walt, 2008 [2]) which we briefly discuss here, is to a large extent universally applicable.

Nevertheless, one may notice that there remains a large scope for possible enrichment of the basic theory. In particular, the space of generalized solutions may depend on the nonlinear partial differential operator (8). Moreover, there is no differential structure on the space of generalized functions associated with the operator $\mathbf{T}(x, D)$. The aim of this paper is to resolve these issues. This is achieved by setting up appropriate uniform convergence spaces (Beattie and Butzmann, 2002), somewhat in the spirit of Sobolev, which do not depend on the particular operator $\mathbf{T}(x, D)$.

The paper is organized as follows. Section 2 introduces the relevant spaces of functions upon which the appropriate spaces of generalized functions are constructed. In Section 3 we discuss the approximation results underlying the Order Completion Method, and introduce a suitable condition on the system of nonlinear PDEs that allows the existence of generalized solutions in appropriate spaces of generalized functions. These results are then applied in Section 4 where we prove the existence of generalized solutions. The structure of the spaces of generalized functions, and that of the generalized functions that are their elements, is discussed in Section 5.

2. Function Spaces and Their Completions

Recall (van der Walt, 2008 [1]), (Anguelov et. al, 2006), (Dilworth, 1950) that an extended real valued function $u : \Omega \rightarrow \mathbb{R}$ is normal lower semi-continuous if

$$(I \circ S)(u)(x) = u(x), \quad x \in \Omega \quad (9)$$

where

$$I(u)(x) = \sup\{\inf\{u(y) : y \in \Omega, \|x - y\| < \delta\} : \delta > 0\} \quad (10)$$

and

$$S(u)(x) = \inf\{\sup\{u(y) : y \in \Omega, \|x - y\| < \delta\} : \delta > 0\} \quad (11)$$

are the the Lower and Upper Baire Operators respectively, see (Anguelov, 2004) and (Baire, 1905). These operators, as well as their composition, are monotone with respect to the pointwise ordering of functions $u : \Omega \rightarrow \mathbb{R}$, and idempotent. A normal lower semi-continuous function is said to be nearly finite if

$$\{x \in \Omega : u(x) \in \mathbb{R}\} \text{ open and dense}$$

The space of all nearly finite normal lower semi-continuous on Ω is denoted by $\mathcal{NL}(\Omega)$. These functions satisfy the following well know property of continuous real valued functions, namely,

$$\begin{aligned} & \forall u, v \in \mathcal{NL}(\Omega) : \\ & \forall D \subseteq \Omega \text{ dense} : \\ & u(x) \leq v(x), \quad x \in D \Rightarrow u(x) \leq v(x), \quad x \in \Omega \end{aligned} \quad (12)$$

Moreover, for every $u \in \mathcal{NL}(\Omega)$ there is a set $B \subset \Omega$ of first Baire Category such that $u \in \mathcal{C}(\Omega \setminus B)$.

Clearly, each function that is continuous is also normal lower semi-continuous. For $0 \leq l \leq \infty$, the subspace of $\mathcal{NL}(\Omega)$ consisting of functions that are continuous with

continuous partial derivatives up to order l on some open and sense subset of Ω is denoted $\mathcal{ML}^l(\Omega)$. That is,

$$\mathcal{ML}^l(\Omega) = \left\{ u \in \mathcal{NL}(\Omega) \mid \begin{array}{l} \exists \Gamma \subset \Omega \text{ closed nowhere dense :} \\ u \in \mathcal{C}^l(\Omega \setminus \Gamma) \end{array} \right\} \quad (13)$$

The space $\mathcal{NL}(\Omega)$, ordered in a pointwise way, is a fully distributive lattice (van der Walt, 2008 [1]), and contains $\mathcal{ML}^0(\Omega)$ as a sublattice. Therefore (van der Walt, 2006) the order convergence of sequences (Luxemburg and Zaanen, 1971) on $\mathcal{ML}^0(\Omega)$, which is defined through

$$(u_n) \text{ order converges to } u \Leftrightarrow \left(\begin{array}{l} \exists (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ 1) \lambda_n \leq \lambda_{n+1} \leq u_{n+1} \leq \mu_{n+1} \leq \mu_n, n \in \mathbb{N} \\ 2) \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\} \end{array} \right) \quad (14)$$

is induced by a convergence structure (Beattie and Butzmann, 2002). In fact, the uniform convergence structure \mathcal{J}_o (van der Walt, 2008 [1]) induces the order convergence of sequences, and is defined as follows.

DEFINITION 1. A filter \mathcal{U} on $\mathcal{ML}^0(\Omega) \times \mathcal{ML}^0(\Omega)$ belongs to the family \mathcal{J}_o whenever there exists $k \in \mathbb{N}$ such that, for every $i = 1, \dots, k$, there exists nonempty order intervals $(I_n^i)_{n \in \mathbb{N}}$ such that following conditions are satisfied:

- 1) $I_n^i \supseteq I_{n+1}^i$ for every $n \in \mathbb{N}$.
- 2) If $V \subseteq \Omega$ is open, then $\bigcap_{n \in \mathbb{N}} I_n^i|_V = \emptyset$, or there exists $u_i \in \mathcal{ML}^0(V)$ such that $\bigcap_{n \in \mathbb{N}} I_n^i|_V = \{u_i\}$.
- 3) $(\mathcal{I}^1 \times \mathcal{I}^1) \cap \dots \cap (\mathcal{I}^k \times \mathcal{I}^k) \subset \mathcal{U}$

Here $\mathcal{I}^i = [\{I_n^i : n \in \mathbb{N}\}]$ and $I_n^i|_V$ consists of all functions $u \in I_n^i$, restricted to V .

The uniform convergence structure \mathcal{J}_o is uniformly Hausdorff and first countable. Moreover, a filter \mathcal{F} on $\mathcal{ML}^0(\Omega)$ converges to $u \in \mathcal{ML}^0(\Omega)$ with respect to \mathcal{J}_o if and only if

$$\begin{array}{l} \exists (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ 1) \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n, n \in \mathbb{N} \\ 2) \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\} \\ 3) [\{\lambda_n, \mu_n\} : n \in \mathbb{N}] \subseteq \mathcal{F} \end{array} \quad (15)$$

The completion of the uniform convergence space $\mathcal{ML}^0(\Omega)$ is obtained as the space $\mathcal{NL}(\Omega)$ equipped with an appropriate uniform convergence structure \mathcal{J}_o^\sharp , see (van der Walt, 2008 [1]). In particular, the uniform convergence structure \mathcal{J}_o^\sharp induces the order convergence structure (15).

The usual partial differential operators on $\mathcal{C}^l(\Omega)$, with $l \geq 1$, may be extended to $\mathcal{ML}^l(\Omega)$ through

$$\mathcal{D}^\alpha : \mathcal{ML}^l(\Omega) \ni u \mapsto (I \circ S)(\mathcal{D}^\alpha u) \in \mathcal{ML}^0(\Omega) \quad (16)$$

Therefore, somewhat in the spirit of Sobolev, we equip the space $\mathcal{ML}^l(\Omega)$, where $l \geq 1$, with the initial uniform convergence structure \mathcal{J}_D with respect to the family of mappings

$$(\mathcal{D}^\alpha : \mathcal{ML}^l(\Omega) \rightarrow \mathcal{ML}^0(\Omega))_{|\alpha| \leq l} \quad (17)$$

That is, for any filter \mathcal{U} on $\mathcal{ML}^l(\Omega) \times \mathcal{ML}^l(\Omega)$, we have

$$\mathcal{U} \in \mathcal{J}_D \Leftrightarrow \left(\forall |\alpha| \leq l \quad (\mathcal{D}^\alpha \times \mathcal{D}^\alpha)(\mathcal{U}) \in \mathcal{J}_o \right) \quad (18)$$

Since the family of mappings (17) separates the elements of $\mathcal{ML}^l(\Omega)$, that is,

$$\begin{aligned} & \forall u, v \in \mathcal{ML}^l(\Omega) : \\ & \exists |\alpha| \leq l : \quad \mathcal{D}^\alpha u \neq \mathcal{D}^\alpha v \end{aligned} \quad ,$$

it follows that \mathcal{J}_D is uniformly Hausdorff. A filter \mathcal{F} on $\mathcal{ML}^l(\Omega)$ is a Cauchy filter if and only if $\mathcal{D}^\alpha(\mathcal{F})$ is a Cauchy filter in $\mathcal{ML}^0(\Omega)$ for each $|\alpha| \leq l$. In particular, a filter \mathcal{F} on $\mathcal{ML}^l(\Omega)$ converges to $u \in \mathcal{ML}^l(\Omega)$ if and only if $\mathcal{D}^\alpha(\mathcal{F})$ converges to $\mathcal{D}^\alpha u$ in $\mathcal{ML}^0(\Omega)$ for each $|\alpha| \leq l$. In view of the results (van der Walt, 2007 [2]) on the completion of uniform convergence spaces, the completion of $\mathcal{ML}^l(\Omega)$ is realized as a subspace of $\mathcal{NL}(\Omega)^M$, for an appropriate $M \in \mathbb{N}$. In analogy with the case $l = 0$ we denote the completion of $\mathcal{ML}^l(\Omega)$ by $\mathcal{NL}^l(\Omega)$. The structure of the space $\mathcal{NL}^l(\Omega)$ and its elements will be discussed in Section 5.

With a system of PDEs of the form (6) we may associate a mapping

$$\mathbf{T} : \mathcal{ML}^m(\Omega)^K \rightarrow \mathcal{ML}^0(\Omega)^K \quad (19)$$

Indeed, we can write the system (6) componentwise as

$$\begin{aligned} T_1(x, D) \mathbf{u}(x) &= f_1(x) \\ &\vdots \quad \quad \quad \vdots \\ T_j(x, D) \mathbf{u}(x) &= f_j(x) \\ &\vdots \quad \quad \quad \vdots \\ T_K(x, D) \mathbf{u}(x) &= f_K(x) \end{aligned} \quad (20)$$

where, for each $j = 1, \dots, K$ the component $T_j(x, D)$ of $\mathbf{T}(x, D)$ is defined through the component F_j of the mapping \mathbf{F} by

$$T_j(x, D) \mathbf{u}(x) = F_j(x, u_1(x), \dots, u_K(x), \dots, \mathcal{D}^\alpha u_i(x), \dots), |\alpha| \leq m \text{ and } i = 1, \dots, K \quad (21)$$

The mappings (21) are then extended to $\mathcal{ML}^m(\Omega)$ through

$$T_j : \mathcal{ML}^m(\Omega)^K \ni \mathbf{u} \mapsto (I \circ S)(F_j(\cdot, u_1, \dots, u_K, \dots, \mathcal{D}^\alpha u_i, \dots)) \in \mathcal{ML}^0(\Omega), \quad (22)$$

yielding the components of the mapping (19). An equivalence relation is induced on $\mathcal{ML}^m(\Omega)^K$ by the mapping \mathbf{T} through

$$\begin{aligned} & \forall \mathbf{u}, \mathbf{v} \in \mathcal{ML}^m(\Omega)^K : \\ & \mathbf{u} \sim_{\mathbf{T}} \mathbf{v} \Leftrightarrow \mathbf{T}\mathbf{u} = \mathbf{T}\mathbf{v} \end{aligned} \quad (23)$$

The resulting quotient space $\mathcal{ML}^m(\Omega)^K / \sim_{\mathbf{T}}$ is denoted $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$. With the mapping (19) we may associate in a canonical way an *injective* mapping

$$\hat{\mathbf{T}} : \mathcal{ML}_{\mathbf{T}}^m(\Omega) \rightarrow \mathcal{ML}^0(\Omega)^K \quad (24)$$

so that the diagram

$$\begin{array}{ccc}
\mathcal{ML}^m(\Omega)^K & \xrightarrow{\mathbf{T}} & \mathcal{ML}^0(\Omega)^K \\
\downarrow q_{\mathbf{T}} & & \downarrow id \\
\mathcal{ML}_{\mathbf{T}}^m(\Omega) & \xrightarrow{\hat{\mathbf{T}}} & \mathcal{ML}^0(\Omega)^K
\end{array}$$

commutes. Here $q_{\mathbf{T}}$ denotes the canonical quotient map associated with the equivalence relation (23), and id is the identity on $\mathcal{ML}^0(\Omega)^K$.

The product spaces $\mathcal{ML}^0(\Omega)^K$ and $\mathcal{ML}^m(\Omega)^K$ will carry, naturally, the product uniform convergence structures \mathcal{J}_o^K and \mathcal{J}_D^K , respectively. That is,

$$\mathcal{U} \in \mathcal{J}_o^K \Leftrightarrow \left(\begin{array}{l} \forall i = 1, \dots, K : \\ (\pi_i \times \pi_i)(\mathcal{U}) \in \mathcal{J}_o \end{array} \right) \quad (25)$$

and

$$\mathcal{U} \in \mathcal{J}_D^K \Leftrightarrow \left(\begin{array}{l} \forall i = 1, \dots, K : \\ (\pi_i \times \pi_i)(\mathcal{U}) \in \mathcal{J}_D \end{array} \right) \quad (26)$$

Here π_i denotes the projection on the i th coordinate. The completion of $\mathcal{ML}^0(\Omega)^K$ is $\mathcal{NL}(\Omega)^K$ equipped with the product uniform convergence structure induced by \mathcal{J}_o^\sharp . Similarly, the completion of $\mathcal{ML}^m(\Omega)^K$ is $\mathcal{NL}^m(\Omega)^K$.

The space $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ carries the initial uniform convergence structure $\mathcal{J}_{\hat{\mathbf{T}}}$ with respect to the mapping $\hat{\mathbf{T}}$. That is,

$$\mathcal{U} \in \mathcal{J}_{\hat{\mathbf{T}}} \Leftrightarrow \hat{\mathbf{T}}(\mathcal{U}) \in \mathcal{J}_o^K \quad (27)$$

Since $\hat{\mathbf{T}}$ is *injective*, it follows that $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ is uniformly isomorphic to the subspace $\hat{\mathbf{T}}(\mathcal{ML}_{\mathbf{T}}^m(\Omega))$ of $\mathcal{ML}^0(\Omega)^K$. In particular, $\hat{\mathbf{T}}$ is a uniformly continuous embedding. In view of the general results on the completion of uniform convergence spaces (van der Walt, 2007 [2]), the completion $\mathcal{NL}_{\mathbf{T}}^m(\Omega)$ of $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ is homeomorphic to a subspace of $\mathcal{NL}(\Omega)^K$. Indeed, $\hat{\mathbf{T}}$ extends to a uniformly continuous embedding of $\mathcal{NL}_{\mathbf{T}}^m(\Omega)$ into $\mathcal{NL}(\Omega)^K$.

3. Approximation Results

In this section we consider the approximation results underlying the Order Completion Method (Oberuggenberger and Rosinger, 1994) and the pseudo-topological version of that theory developed in (van der Walt, 2008 [2]). In this regard we again consider a system of nonlinear PDEs of the form (6) through (8). Recall (van der Walt, 2008 [2]) that, subject to the mild assumption (7), we obtain the following *local* approximation result. We include the proof as an illustration of the technique used. Moreover, it serves to clarify the arguments that lead to a refinement of these results.

PROPOSITION 2. *Consider a system of PDEs of the form (6) through (8) that also satisfies (7). Then*

$$\begin{aligned} & \forall x_0 \in \Omega : \\ & \forall \epsilon > 0 : \\ & \exists \delta > 0, P_1, \dots, P_K \text{ polynomial in } x \in \mathbb{R}^n : \\ & x \in B(x_0, \delta) \cap \Omega, 1 \leq i \leq K \Rightarrow f_i(x) - \epsilon < T_i(x, D) \mathbf{P}(x) < f_i(x) \end{aligned} \quad (28)$$

Here \mathbf{P} is the K -dimensional vector valued function with components P_1, \dots, P_K .

Proof. For any $x_0 \in \Omega$ and $\epsilon > 0$ small enough it follows by (7) that there exist

$$\xi_{i\alpha} \in \mathbb{R} \text{ with } i = 1, \dots, K \text{ and } |\alpha| \leq m$$

such that

$$F_i(x_0, \dots, \xi_{i\alpha}, \dots) = f_i(x_0) - \frac{\epsilon}{2}$$

Now take P_1, \dots, P_K polynomials in $x \in \mathbb{R}^n$ that satisfy

$$D^\alpha P_i(x_0) = \xi_{i\alpha} \text{ for } i = 1, \dots, K \text{ and } |\alpha| \leq m$$

Then it is clear that

$$T_i(x, D) \mathbf{P}(x_0) - f_i(x_0) = -\frac{\epsilon}{2}$$

where \mathbf{P} is the K -dimensional vector valued function on \mathbb{R}^n with components P_1, \dots, P_K . Hence (28) follows by the continuity of the f_i and the F_i . ■

It should be observed that, in contradistinction to the usual functional analytic methods, the local *lower solution* in Proposition 2 is constructed in a particularly simple way. Indeed, it is obtained by nothing but a straightforward application of the continuity of the mapping \mathbf{F} . Using exactly these same techniques, one may prove the existence of the corresponding approximate *upper solutions*.

PROPOSITION 3. *Consider a system of PDEs of the form (6) through (8) that also satisfies (7). Then*

$$\begin{aligned} & \forall x_0 \in \Omega : \\ & \forall \epsilon > 0 : \\ & \exists \delta > 0, P_1, \dots, P_K \text{ polynomial in } x \in \mathbb{R}^n : \\ & x \in B(x_0, \delta) \cap \Omega, 1 \leq i \leq K \Rightarrow f_i(x) < T_i(x, D) \mathbf{P}(x) < f_i(x) + \epsilon \end{aligned}$$

Here \mathbf{P} is the K -dimensional vector valued function with components P_1, \dots, P_K .

The *global* approximations, corresponding to the local approximation constructed in Propositions 2 and 3, may be formulated as follows.

THEOREM 4. *Consider a system of PDEs of the form (6) through (8) that also satisfies (7). For every $\epsilon > 0$ there exists a closed nowhere dense set $\Gamma_\epsilon \subset \Omega$, and functions $\mathbf{U}_\epsilon, \mathbf{V}_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)^K$ with components $U_{\epsilon,1}, \dots, U_{\epsilon,K}$ and $V_{\epsilon,1}, \dots, V_{\epsilon,K}$ respectively, such that*

$$f_i(x) - \epsilon < T_i(x, D) \mathbf{U}_\epsilon(x) < f_i(x) < T_i(x, D) \mathbf{V}_\epsilon(x) < f_i(x) + \epsilon, x \in \Omega \setminus \Gamma_\epsilon \quad (29)$$

Once again, and as was mentioned in connection with Proposition 2, the approximation result above is based *solely* on the existence of a compact tiling of any open subset Ω of \mathbb{R}^n , the properties of compact subsets of \mathbb{R}^n and the continuity of usual real valued functions, see for instance (van der Walt, 2008 [2], Theorem 4). Hence it makes no use of so called *advanced mathematics*. In particular, techniques from functional analysis are not used at all. Instead, the relevant techniques belong rather to the classical theory of real functions.

As an immediate application of Theorem 4, we may construct a sequence (\mathbf{u}_n) in $\mathcal{ML}^m(\Omega)^K$ so that $(\mathbf{T}\mathbf{u}_n)$ converges to \mathbf{f} in $\mathcal{ML}^0(\Omega)^K$. However, Theorem 4 makes no claim concerning the convergence of the sequence (\mathbf{u}_n) , or lack thereof, in $\mathcal{ML}^m(\Omega)^K$. Indeed, assuming only that (7) is satisfied, it is possible, in almost all cases of applicative interest, to construct a sequence (\mathbf{U}_n) that satisfies Theorem 4, and is unbounded on every neighborhood of every point of Ω . This follows easily from the fact that, in general, for a fixed $x_0 \in \Omega$, the sets

$$\{\xi \in \mathbb{R}^M : \mathbf{F}(x_0, \xi) = \mathbf{f}(x_0)\}$$

may be unbounded.

In view of the above remarks, it appears that a stronger assumption than (7) may be required in order to construct generalized solutions to (6) in $\mathcal{NL}^m(\Omega)^K$. When formulating such an appropriate condition on the system of PDEs (6), one should keep in mind that the Order Completion Method (Oberguggenberger and Rosinger, 1994), and in particular the pseudo-topological version of the theory developed in (van der Walt, 2008 [1]) and (van der Walt, 2008 [2]), is based on some basic topological processes, namely, the completion of uniform convergence spaces, and the simple condition (7), which is formulated entirely in terms of the usual real mappings \mathbf{F} and \mathbf{f} . In particular, (7) does not involve any topological structures on function spaces, or mappings on such spaces. Furthermore, other than the mere continuity of the mapping \mathbf{F} , (7) places no restriction on the *type* of equation treated. As such, it is then clear that any further assumptions that we may wish to impose on the system of equations (6) in order to obtain generalized solutions in $\mathcal{NL}^m(\Omega)^K$ should involve only basic topological properties of the mapping \mathbf{F} , and should not involve any restrictions on the type of equations.

In formulating such a condition on the system of PDEs (6) that will ensure the existence of a generalized solution in $\mathcal{NL}^m(\Omega)^K$, it is helpful to first understand more completely the role of the condition (7) in the proof of the local approximation result Proposition 2. In particular, and as is clear from the proof of Proposition 2, the condition (7) relates to the continuity of the mapping \mathbf{F} . Furthermore, and as has already been mentioned, the approximations constructed in Proposition 2 and Theorem 4 concern only convergence in the target space of the operator \mathbf{T} associated with (6). Our interest here lies in constructing suitable approximations in the domain of \mathbf{T} , and as such, properties of the inverse of the mapping \mathbf{F} may prove to be particularly useful. In view of these remarks, we introduce the following condition.

$$\begin{aligned} & \forall x_0 \in \Omega : \\ & \exists \xi(x_0) \in \mathbb{R}^M : \\ & \exists V \in \mathcal{V}_{x_0}, W \in \mathcal{V}_{\xi(x_0)} : \\ & \quad 1) \mathbf{F}(x_0, \xi(x_0)) = \mathbf{f}(x_0) : \\ & \quad 2) \mathbf{F} : V \times W \rightarrow \mathbb{R}^K \text{ open} \end{aligned} \tag{30}$$

Note that the condition (30) above, although more restrictive than (7), allows for the treatment of a large class of equations. In particular, each equation of the form

$$D_t \mathbf{u}(x, t) + \mathbf{G}(x, t, \mathbf{u}(x, t), \dots, D_x^\alpha \mathbf{u}(x, t), \dots) = \mathbf{f}(x, t)$$

with the mapping \mathbf{G} merely continuous, satisfies (30). Indeed, the mapping \mathbf{F} that defines the equation through (6) is both open and surjective. Other classes of equations that satisfy (30) can be easily exhibited.

4. Existence of Generalized Solutions

The basic existence result for the Order Completion Method (van der Walt, 2008 [2]) is an application of the global approximation result in Theorem 4, and the commutative diagram

$$\begin{array}{ccc}
 \mathcal{ML}_{\mathbf{T}}^m(\Omega) & \xrightarrow{\hat{\mathbf{T}}} & \mathcal{ML}^0(\Omega)^K \\
 \downarrow \phi & & \downarrow \varphi \\
 \mathcal{NL}_{\mathbf{T}}(\Omega) & \xrightarrow{\hat{\mathbf{T}}^\#} & \mathcal{NL}(\Omega)^K
 \end{array}$$

Here ϕ and φ are the uniformly continuous embeddings associated canonically with the completions $\mathcal{NL}_{\mathbf{T}}(\Omega)$ and $\mathcal{NL}(\Omega)$, and $\hat{\mathbf{T}}^\#$ is the extension of $\hat{\mathbf{T}}$ achieved through uniform continuity. The approximation result, Theorem 4, is used to construct a Cauchy sequence in $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ so that its image under $\hat{\mathbf{T}}$ converges to \mathbf{f} . This delivers the *existence* of a solution to the generalized equation

$$\hat{\mathbf{T}}^\# \mathbf{U}^\# = \mathbf{f} \quad (31)$$

which we interpret as a generalized solution to (6). Moreover, since $\hat{\mathbf{T}}$ is a uniformly continuous embedding, so is its extension $\hat{\mathbf{T}}^\#$ and hence the solution to (31) is *unique*.

THEOREM 5. *For every $\mathbf{f} \in \mathcal{C}^0(\Omega)^K$ that satisfies (7), there exists a unique $\mathbf{U}^\# \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ so that*

$$\hat{\mathbf{T}}^\# \mathbf{U}^\# = \mathbf{f} \quad (32)$$

The aim of this section is to obtain the existence of generalized solutions to (6) in the space $\mathcal{NL}^m(\Omega)^K$. In order to formulate an extended version of the equation (6) in this setting, the partial differential operator \mathbf{T} must be extended to the completion of $\mathcal{ML}^m(\Omega)^K$. The mapping \mathbf{T} must therefore be uniformly continuous with respect to the uniform convergence structures on $\mathcal{ML}^m(\Omega)^K$ and $\mathcal{ML}^0(\Omega)^K$. The proof of this result is deferred to the appendix.

THEOREM 6. *The mapping*

$$\mathbf{T}: \mathcal{ML}^m(\Omega)^K \rightarrow \mathcal{ML}^0(\Omega)^K$$

defined in (19) to (22) is uniformly continuous.

In view of Theorem 6 the mapping \mathbf{T} extends uniquely to a uniformly continuous mapping

$$\mathbf{T}^\sharp : \mathcal{NL}^m(\Omega)^K \rightarrow \mathcal{NL}(\Omega)^K \quad (33)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega)^K & \xrightarrow{\mathbf{T}} & \mathcal{ML}^0(\Omega)^K \\ \psi \downarrow & & \downarrow \varphi \\ \mathcal{NL}^m(\Omega)^K & \xrightarrow{\mathbf{T}^\sharp} & \mathcal{NL}(\Omega)^K \end{array}$$

commutes, with ψ and φ the uniformly continuous embeddings associated with the completions of $\mathcal{ML}^m(\Omega)^K$ and $\mathcal{ML}^0(\Omega)^K$, respectively. Therefore we are justified in formulating the generalized equation

$$\mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \quad (34)$$

where the unknown \mathbf{u}^\sharp ranges over $\mathcal{NL}^m(\Omega)^K$.

THEOREM 7. *For every $\mathbf{f} \in \mathcal{C}^0(\Omega)^K$ so that the system of PDEs (6) satisfies (30), there is some $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ so that*

$$\mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \quad (35)$$

Proof. Set

$$\Omega = \bigcup_{\nu \in \mathbb{N}} C_\nu \quad (36)$$

where, for $\nu \in \mathbb{N}$, the compact sets C_ν are n -dimensional intervals

$$C_\nu = [a_\nu, b_\nu] \quad (37)$$

with $a_\nu = (a_{\nu,1}, \dots, a_{\nu,n})$, $b_\nu = (b_{\nu,1}, \dots, b_{\nu,n}) \in \mathbb{R}^n$ and $a_{\nu,j} \leq b_{\nu,j}$ for every $j = 1, \dots, n$. We also assume that the C_ν , with $\nu \in \mathbb{N}$ are locally finite, that is,

$$\begin{aligned} & \forall x \in \Omega : \\ & \exists V \subseteq \Omega \text{ a neighborhood of } x : \\ & \quad \{\nu \in \mathbb{N} : C_\nu \cap V \neq \emptyset\} \text{ is finite} \end{aligned} \quad (38)$$

We also assume that the interiors of C_ν , with $\nu \in \mathbb{N}$, are pairwise disjoint.

Let C_ν be arbitrary but fixed. In view of (30) and the continuity of \mathbf{f} , we have

$$\begin{aligned} & \forall x_0 \in C_\nu : \\ & \exists \xi(x_0) \in \mathbb{R}^M, \mathbf{F}(x_0, \xi(x_0)) = \mathbf{f}(x_0) : \\ & \exists \delta, \epsilon > 0 : \\ & \quad 1) \{(x, \mathbf{f}(x)) : \|x - x_0\| < \delta\} \subset \text{int} \left\{ (x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} \|x - x_0\| < \delta \\ \|\xi - \xi(x_0)\| < \epsilon \end{array} \right. \right\} \\ & \quad 2) \mathbf{F} : B_\delta(x_0) \times B_{2\epsilon}(\xi(x_0)) \rightarrow \mathbb{R}^K \text{ open} \end{aligned} \quad (39)$$

For each $x_0 \in C_\nu$, fix $\xi(x_0) \in \mathbb{R}^M$ in (39). Since C_ν is compact, it follows from (39) that

$$\begin{aligned} & \exists \delta > 0 : \\ & \forall x_0 \in C_\nu : \\ & \exists \epsilon > 0 : \\ & 1) \{ (x, \mathbf{f}(x)) : \|x - x_0\| < \delta \} \subset \text{int} \left\{ (x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} \|x - x_0\| < \delta \\ \|\xi - \xi(x_0)\| < \epsilon \end{array} \right. \right\} \\ & 2) \mathbf{F} : B_\delta(x_0) \times B_{2\epsilon}(\xi(x_0)) \rightarrow \mathbb{R}^K \text{ open} \end{aligned} \quad (40)$$

Subdivide C_ν into n -dimensional intervals $I_{\nu,1}, \dots, I_{\nu,\mu_\nu}$ with diameter not exceeding δ such that their interiors are pairwise disjoint. If $a_{\nu,j}$ with $j = 1, \dots, \mu_\nu$ is the center of the interval $I_{\nu,j}$ then by (40) we have

$$\begin{aligned} & \forall j = 1, \dots, \mu_\nu : \\ & \exists \epsilon_{\nu,j} > 0 : \\ & 1) \{ (x, \mathbf{f}(x)) : x \in I_{\nu,j} \} \subset \text{int} \left\{ (x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} x \in I_{\nu,j} \\ \|\xi - \xi(a_{\nu,j})\| < \epsilon_{\nu,j} \end{array} \right. \right\} \\ & 2) \mathbf{F} : I_{\nu,j} \times B_{2\epsilon_{\nu,j}}(\xi(a_{\nu,j})) \rightarrow \mathbb{R}^K \text{ open} \end{aligned} \quad (41)$$

Take $\gamma > 0$ arbitrary but fixed. In view of Proposition 2 and (41), we have

$$\begin{aligned} & \forall x_0 \in I_{\nu,j} : \\ & \exists \mathbf{U}_{x_0} = \mathbf{U} \in \mathcal{C}^m(\mathbb{R}^n)^K : \\ & \exists \delta = \delta_{x_0} > 0 : \\ & \forall i = 1, \dots, K : \\ & x \in B_\delta(x_0) \cap I_{\nu,j} \Rightarrow \begin{pmatrix} 1) D^\alpha U_i(x) \in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})), |\alpha| \leq m \\ 2) f_i(x) - \gamma < T_i(x, D) \mathbf{U}(x) < f_i(x) \end{pmatrix} \end{aligned}$$

As above, we may subdivide $I_{\nu,j}$ into pairwise disjoint, n -dimensional intervals $J_{\nu,j,1}, \dots, J_{\nu,j,\mu_{\nu,j}}$ so that for $k = 1, \dots, \mu_{\nu,j}$ we have

$$\begin{aligned} & \exists \mathbf{U}^{\nu,j,k} = \mathbf{U} \in \mathcal{C}^m(\mathbb{R}^n)^K : \\ & \forall i = 1, \dots, K : \\ & x \in J_{\nu,j,k} \Rightarrow \begin{pmatrix} 1) D^\alpha U_i(x) \in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})), |\alpha| \leq m \\ 2) f_i(x) - \gamma < T_i(x, D) \mathbf{U}(x) < f_i(x) \end{pmatrix} \end{aligned} \quad (42)$$

Set

$$\mathbf{V}_1 = \sum_{\nu \in \mathbb{N}} \left(\sum_{j=1}^{\mu_\nu} \left(\sum_{k=1}^{\mu_{\nu,j}} \chi_{J_{\nu,j,k}} \mathbf{U}_{\nu,j,k} \right) \right)$$

where $\chi_{J_{\nu,j,k}}$ is the characteristic function of $J_{\nu,j,k}$, and

$$\Gamma_1 = \Omega \setminus \left(\bigcup_{\nu \in \mathbb{N}} \left(\bigcup_{j=1}^{\mu_\nu} \left(\bigcup_{k=1}^{\mu_{\nu,j}} \text{int} J_{\nu,j,k} \right) \right) \right). \quad (43)$$

Then Γ_1 is closed nowhere dense, and $\mathbf{V}_1 \in \mathcal{C}^m(\Omega \setminus \Gamma_1)^K$. In view of (42) we have, for each $i = 1, \dots, K$

$$f_i(x) - \gamma < T_i(x, D) \mathbf{V}_1(x) < f_i(x), \quad x \in \Omega \setminus \Gamma_1$$

Furthermore, for each $\nu \in \mathbb{N}$, for each $j = 1, \dots, \mu_\nu$, each $k = 1, \dots, \mu_{\nu,j}$, each $|\alpha| \leq m$ and every $i = 1, \dots, K$ we have

$$x \in \text{int} J_{\nu,j,k} \Rightarrow \xi_i^\alpha(a_{\nu,j}) - \epsilon < D^\alpha V_{1,i}(x) < \xi_i^\alpha(a_j) + \epsilon$$

Therefore the functions $\lambda_{1,i}^\alpha, \mu_{1,i}^\alpha \in \mathcal{C}^0(\Omega \setminus \Gamma_1)$ defined as

$$\lambda_{1,i}^\alpha(x) = \xi_i^\alpha(a_j) - 2\epsilon_{\nu,j} \text{ if } x \in \text{int}I_{\nu,j}$$

and

$$\mu_{1,i}^\alpha(x) = \xi_i^\alpha(a_j) + 2\epsilon_{\nu,j} \text{ if } x \in \text{int}I_{\nu,j}$$

satisfies

$$\lambda_{1,i}^\alpha(x) < D^\alpha V_{1,i}(x) < \mu_{1,i}^\alpha(x), x \in \Omega \setminus \Gamma_1$$

and

$$\mu_{1,i}^\alpha(x) - \lambda_{1,i}^\alpha(x) < 4\epsilon_{\nu,j}, x \in \text{int}I_{\nu,j}$$

Applying (41), and proceeding in a fashion similar as above, we may construct, for each $n \in \mathbb{N}$ such that $n > 1$, a closed nowhere dense set $\Gamma_n \subset \Omega$, a function $\mathbf{V}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)^K$ and functions $\lambda_{n,i}^\alpha, \mu_{n,i}^\alpha \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$ so that, for each $i = 1, \dots, K$ and $|\alpha| \leq m$

$$f_i(x) - \frac{\gamma}{n} < T_i(x, D) \mathbf{V}_n(x) < f_i(x), x \in \Omega \setminus (\Gamma_n \cup \Gamma_{n-1}). \quad (44)$$

Furthermore,

$$\lambda_{n-1,i}^\alpha(x) < \lambda_{n,i}^\alpha(x) < D^\alpha V_{n,i}(x) < \mu_{n,i}^\alpha(x) < \mu_{n-1,i}^\alpha(x), x \in \Omega \setminus \Gamma_n \quad (45)$$

and

$$\mu_{n,i}^\alpha(x) - \lambda_{n,i}^\alpha(x) < \frac{4\epsilon_{\nu,j}}{n}, x \in (\text{int}I_{\nu,j}) \cap (\Omega \setminus \Gamma_n). \quad (46)$$

For each $n \in \mathbb{N}$ and $i = 1, \dots, K$ set $v_{n,i} = (I \circ S)(V_{n,i})$. Furthermore, for each $|\alpha| \leq m$ set $\bar{\lambda}_{n,i}^\alpha = (I \circ S)(\lambda_{n,i}^\alpha)$ and $\bar{\mu}_{n,i}^\alpha = (I \circ S)(\mu_{n,i}^\alpha)$. Then, in view of (44) we have

$$\begin{aligned} \forall n \in \mathbb{N} : \\ \forall i = 1, \dots, K : \\ f_i - \frac{\gamma}{n} \leq T_i \mathbf{v}_n \leq f_i \end{aligned}$$

and from (45) it follows that

$$\begin{aligned} \forall n \in \mathbb{N} : \\ \forall i = 1, \dots, K : \\ \forall |\alpha| \leq m : \\ \bar{\lambda}_{n,i}^\alpha \leq \bar{\lambda}_{n+1,i}^\alpha \leq \mathcal{D}^\alpha v_{n,i} \leq \bar{\mu}_{n+1,i}^\alpha \leq \bar{\mu}_{n,i}^\alpha \end{aligned} .$$

Moreover, in view of (46) it follows that

$$\bar{\mu}_{n,i}^\alpha(x) - \bar{\lambda}_{n,i}^\alpha(x) < \frac{4\epsilon_{\nu,j}}{n}, x \in \text{int}I_{\nu,j}.$$

Therefore (\mathbf{v}_n) is a Cauchy sequence in $\mathcal{ML}^m(\Omega)^K$, and $(\mathbf{T}\mathbf{v}_n)$ converges to \mathbf{f} in $\mathcal{ML}^0(\Omega)^K$. The result now follows by Theorem 6. ■

5. The Structure of Generalized Functions

Recall (van der Walt, 2008 [2]) that, in view of the abstract construction of the completion of a uniform convergence space (Wyler, 1970), the unique solution to the generalized equation (31) may be represented as the equivalence class of Cauchy filters on $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$

$$\left\{ \mathcal{F} \text{ a filter on } \mathcal{ML}_{\mathbf{T}}^m(\Omega) : \hat{\mathbf{T}}(\mathcal{F}) \text{ converges to } \mathbf{f} \right\} \quad (47)$$

That is, it consists of the totality of all filters \mathcal{F} on $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ so that $\hat{\mathbf{T}}(\mathcal{F})$ converges to \mathbf{f} in $\mathcal{ML}^0(\Omega)^K$. Moreover, each classical solution $\mathbf{u} \in \mathcal{C}^m(\Omega)^K$, and also each nonclassical solution $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$ to (6) generates a Cauchy filter in $\mathcal{ML}_{\mathbf{T}}^m(\Omega)$ which belongs to the equivalence class (47). Therefore the generalized solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ is consistent with the usual classical solutions as well as the nonclassical solutions in $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$. This may be represented in the commutative diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega)^K & \xrightarrow{\mathbf{T}} & \mathcal{ML}^0(\Omega)^K \\ & \searrow q_{\mathbf{T}} & \nearrow \hat{\mathbf{T}} \\ & \mathcal{ML}_{\mathbf{T}}^m(\Omega) & \end{array}$$

where $q_{\mathbf{T}}$ is the canonical quotient map associated with the equivalence relation (23).

In view of this diagram it appears that the mapping $\hat{\mathbf{T}}$ is nothing but a *representation* of the usual nonlinear partial differential operator \mathbf{T} . By virtue of the uniform continuity of the mapping \mathbf{T} , we obtain a similar representation for the extended operator \mathbf{T}^\sharp .

$$\begin{array}{ccc} \mathcal{NL}^m(\Omega)^K & \xrightarrow{\mathbf{T}^\sharp} & \mathcal{NL}(\Omega)^K \\ & \searrow q_{\mathbf{T}}^\sharp & \nearrow \hat{\mathbf{T}}^\sharp \\ & \mathcal{NL}_{\mathbf{T}^\sharp}(\Omega) & \end{array}$$

In view of this diagram, every generalized solution $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ to (6) is mapped unto the unique generalized solution in $\mathcal{NL}_{\mathbf{T}^\sharp}(\Omega)$. As such, these two concepts of generalized solution are consistent. Furthermore,

$$\begin{aligned} \forall \mathbf{u}^\sharp, \mathbf{v}^\sharp \in \mathcal{NL}^m(\Omega)^K : \\ \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{T}^\sharp \mathbf{v}^\sharp \Rightarrow q_{\mathbf{T}^\sharp}^\sharp \mathbf{u}^\sharp = q_{\mathbf{T}^\sharp}^\sharp \mathbf{v}^\sharp \end{aligned}$$

so that the space $\mathcal{NL}_{\mathbf{T}^\sharp}(\Omega)$ contains the quotient space $\mathcal{NL}_{\mathbf{T}^\sharp}(\Omega) / \sim_{\mathbf{T}^\sharp}$, where

$$\begin{aligned} \forall \mathbf{u}^\sharp, \mathbf{v}^\sharp \in \mathcal{NL}^m(\Omega)^K : \\ \mathbf{u}^\sharp \sim_{\mathbf{T}^\sharp} \mathbf{v}^\sharp \Leftrightarrow \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{T}^\sharp \mathbf{v}^\sharp \end{aligned}$$

Therefore, the generalized solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ constructed in Theorem 5 may be represented as

$$\mathbf{U}^\sharp = \left\{ \mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K : \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \right\} \quad (48)$$

Regarding the structure of the space $\mathcal{NL}^m(\Omega)$ and its elements, we may recall that the uniform convergence structure \mathcal{I}_D on $\mathcal{ML}^m(\Omega)$ is the initial uniform convergence structure with respect to the family of mappings

$$\left(\mathcal{D}^\alpha : \mathcal{ML}^m(\Omega) \rightarrow \mathcal{ML}^0(\Omega) \right)_{|\alpha| \leq m}$$

As such, the mapping

$$\mathbf{D} : \mathcal{ML}^m(\Omega) \ni u \mapsto (\mathcal{D}^\alpha u)_{|\alpha| \leq m} \in \mathcal{ML}^0(\Omega)^M$$

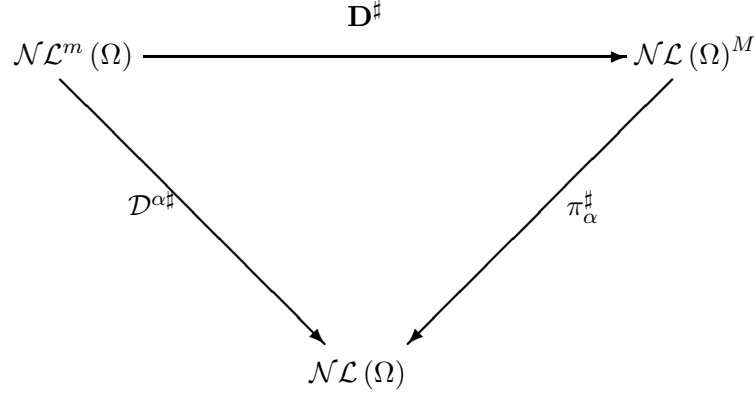
is a uniformly continuous embedding. In particular, for each $|\alpha| \leq m$, the diagram

$$\begin{array}{ccc} & \mathbf{D} & \\ \mathcal{ML}^m(\Omega) & \xrightarrow{\quad} & \mathcal{ML}^0(\Omega)^M \\ & \searrow \mathcal{D}^\alpha \quad \swarrow \pi_\alpha & \\ & \mathcal{ML}^0(\Omega) & \end{array}$$

commutes, with π_α the projection. This diagram amounts to a decomposition of $u \in \mathcal{ML}^m(\Omega)$ into its differential components. In view of the uniform continuity of the mapping \mathbf{D} and its inverse, \mathbf{D} extends to an embedding

$$\mathbf{D}^\sharp : \mathcal{ML}^m(\Omega)^\sharp \rightarrow \mathcal{NL}(\Omega)^M$$

Moreover, since each mapping \mathcal{D}^α is uniformly continuous, one obtains the commutative diagram



where $\mathcal{D}^{\alpha^\sharp}$ is the extension through uniform continuity of the partial differential operator \mathcal{D}^α . Since the mapping \mathbf{D}^\sharp is an embedding, and in view of the commutative diagram above, each generalized function $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)$ may be uniquely represented by its differential components

$$\mathbf{u}^\sharp \mapsto \mathbf{D}^\sharp \mathbf{u}^\sharp = \left(\mathcal{D}^{\alpha^\sharp} \mathbf{u}^\sharp \right)_{|\alpha| \leq m}$$

Moreover, each differential component $\mathcal{D}^{\alpha^\sharp} \mathbf{u}^\sharp$ of \mathbf{u}^\sharp is a nearly finite normal lower semi-continuous function. We note, therefore, that the set of singular points of each $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)$, that is, the set

$$\left\{ x \in \Omega \left| \begin{array}{l} \exists \ |\alpha| \leq m : \\ \mathcal{D}^{\alpha^\sharp} \mathbf{u}^\sharp \text{ not continuous at } x \end{array} \right. \right\}$$

is at most a set of First Baire Category. That is, it is a topologically small set. However, this set may be dense in Ω . Furthermore, such a set may have arbitrarily large positive Lebesgue measure (Oxtoby, 1980).

6. Conclusion

We have established the existence of generalized solutions to a large class of systems of non-linear PDEs. The space of generalized functions that contains the solutions is constructed as the uniform convergence space completion of a space of piecewise smooth functions, and is independent of the particular partial differential equation under consideration. Moreover, the generalized functions may be represented through their differential components as normal lower semi-continuous functions. To what extent the generalized solutions may be interpreted classically, that is, as continuously differentiable functions that satisfy the equations classically, is still an open problem.

In terms of the previous existence and uniqueness results obtained through the Order Completion Method, notably Theorem 5, we may interpret Theorem 7 as a regularity result. In particular, it appears that the generalized solution delivered by Theorem 5 is nothing but the *totality* of all solutions in $\mathcal{NL}^m(\Omega)^K$.

Appendix

This appendix contains the proof of Theorem 6. It is based on the following results which may be found in (Anguelov and van der Walt, 2005), (van der Walt, 2008 [1]) and (van der Walt, 2008 [3]).

PROPOSITION 8. *For any $u \in \mathcal{NL}(\Omega)$ there exists sequences (λ_n) and (μ_n) in $\mathcal{ML}^0(\Omega)$ so that*

$$\begin{aligned} \forall n \in \mathbb{N} : \\ \lambda_n \leq \lambda_{n+1} \leq u \leq \mu_{n+1} \leq \mu_n \end{aligned} \tag{49}$$

and

$$\begin{aligned} \forall x \in \Omega : \\ \sup\{\lambda_n(x) : n \in \mathbb{N}\} = u(x) = \inf\{\mu_n(x) : n \in \mathbb{N}\} \end{aligned}$$

PROPOSITION 9. *Consider a set $U \subset \mathcal{NL}(\Omega)$ so that*

$$\begin{aligned} \exists B \subseteq \Omega \text{ of First Baire Category} : \\ \exists v : \Omega \setminus B \rightarrow \mathbb{R} : \\ x \in \Omega \setminus B \Rightarrow u(x) \leq v(x), u \in U \end{aligned}$$

Then there is some $w \in \mathcal{NL}(\Omega)$ so that

$$\begin{aligned} \forall u \in U : \\ u \leq w \end{aligned}$$

The corresponding statement for sets bounded from below is also true.

PROPOSITION 10. *Let L be a lattice with respect to a given partial order \leq .*

1. *For every $n \in \mathbb{N}$, let the sequence $(u_{m,n})$ in L be bounded and increasing and let*

$$\begin{aligned} u_n &= \sup\{u_{m,n} : m \in \mathbb{N}\}, n \in \mathbb{N} \\ u'_n &= \sup\{u_{m,n} : m = 1, \dots, n\} \end{aligned}$$

If the sequence (u_n) is bounded from above and increasing, and has supremum in L , then the sequence (u'_n) is bounded and increasing and

$$\sup\{u_n : n \in \mathbb{N}\} = \sup\{u'_n : n \in \mathbb{N}\}$$

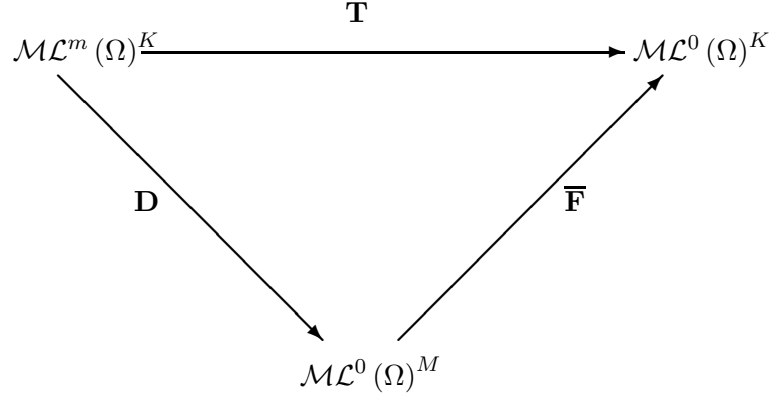
2. *For every $n \in \mathbb{N}$, let the sequence $(v_{m,n})$ in L be bounded and decreasing and let*

$$\begin{aligned} v_n &= \inf\{v_{m,n} : m \in \mathbb{N}\}, n \in \mathbb{N} \\ v'_n &= \inf\{v_{m,n} : m = 1, \dots, n\} \end{aligned}$$

If the sequence (v_n) is bounded from below and decreasing, and has infimum in L , then the sequence (v'_n) is bounded and decreasing and

$$\inf\{v_n : n \in \mathbb{N}\} = \inf\{v'_n : n \in \mathbb{N}\}$$

Proof of Theorem 7. The mapping **T** may be represented through the diagram



where \mathbf{D} maps \mathbf{u} to its vector of derivatives, that is,

$$\mathbf{D} : \mathcal{ML}^m(\Omega)^K \ni \mathbf{u} \mapsto (\mathcal{D}^\alpha u_i)_{|\alpha| \leq m, i \leq K} \in \mathcal{ML}^0(\Omega)^M \quad (50)$$

and $\mathbf{F} = (\overline{F}_i)_{i \leq K}$ is defined componentwise through

$$\overline{F}_i : \mathcal{ML}^0(\Omega)^M \ni \mathbf{u} \mapsto (I \circ S)(F_i(\cdot, u_1, \dots, u_M)) \in \mathcal{ML}^0(\Omega) \quad (51)$$

Clearly the mapping \mathbf{D} is uniformly continuous, so in view of the diagram it suffices to show that $\overline{\mathbf{F}}$ is uniformly continuous with respect to the product uniform convergence structure.

In this regard, we consider sequences of order intervals (I_n^i) in $\mathcal{ML}^0(\Omega)$, where $i = 1, \dots, M$, that satisfies condition 2) of Definition 1. We claim

$$\begin{aligned} & \forall n \in \mathbb{N} : \\ & \exists J_n^1, \dots, J_n^K \subseteq \mathcal{ML}^0(\Omega) \text{ order intervals :} \\ & F_j \left(\prod_{i=1}^M I_n^i \right) \subseteq J_n^j, j = 1, \dots, K \end{aligned} \quad (52)$$

To verify (52), observe that there is a closed nowhere dense set $\Gamma_n \subseteq \Omega$ so that

$$\begin{aligned} & \forall x \in \Omega \setminus \Gamma : \\ & \exists a(x) > 0 : \\ & \forall i = 1, \dots, M : \\ & u \in I_n^i \Rightarrow |u(x)| \leq a(x) \end{aligned} \quad (53)$$

Since $F_j : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}$ is continuous, it follows from (53) that

$$\begin{aligned} & \forall x \in \Omega \setminus \Gamma : \\ & \exists b(x) > 0 : \\ & \left(\forall i = 1, \dots, M : \right. \\ & \quad \left. u_i \in I_n^i \right) \Rightarrow |F_j(x, u_1(x), \dots, u_M(x))| \leq b(x) \end{aligned} \quad (54)$$

Therefore, in view of Proposition 9, our claim (52) holds. In particular, since $\mathcal{NL}(\Omega)$ is Dedekind complete (van der Walt, 2008 [1]), we may set

$$J_n^j = [\lambda_n^j, \mu_n^j]$$

where, for each $n \in \mathbb{N}$ and each $j = 1, \dots, K$

$$\lambda_n^j = \inf \{ \overline{F}_j \mathbf{u} : \mathbf{u} \in \prod_{i=1}^M I_n^i \}$$

and

$$\mu_n^j = \sup\{\overline{F}_j \mathbf{u} : \mathbf{u} \in \prod_{i=1}^M I_n^i\}$$

The sequence (λ_n^j) and (μ_n^j) are increasing and decreasing, respectively. For each $j = 1, \dots, K$ we may consider

$$\sup\{\lambda_n^j : n \in \mathbb{N}\} = u^j \leq v^j = \inf\{\mu_n^j : n \in \mathbb{N}\}$$

We claim that $u^j = v^j$. To see this, we note that for each $j = 1, \dots, K$ there is some $w^j \in \mathcal{NL}(\Omega)$ so that

$$\sup\{l_n^j : n \in \mathbb{N}\} = w^j = \inf\{u_n^j : n \in \mathbb{N}\}$$

where $I_n^j = [l_n^j, u_n^j]$. Applying Lemma 11 and the continuity of F_j our claim is verified. Applying Propositions 8 and 9 we obtain sequence (\overline{T}_n^j) of order intervals in $\mathcal{ML}^0(\Omega)$ that satisfies condition 2) of Definition 1 and

$$\overline{F}_j \left(\prod_{i=1}^M I_n^i \right) \subseteq \overline{T}_n^j$$

This completes the proof. ■

The proof also relies on the following lemma.

LEMMA 11. *Consider a decreasing sequence (u) in $\mathcal{NL}(\Omega)$ that satisfies*

$$u = \inf\{u_n : n \in \mathbb{N}\} \in \mathcal{NL}(\Omega)$$

Then the follows holds:

$$\begin{aligned} & \forall \epsilon > 0 : \\ & \exists \Gamma_\epsilon \subseteq \Omega \text{ closed nowhere dense} : \\ & x \in \Omega \setminus \Gamma_\epsilon \Rightarrow \left(\exists N_\epsilon \in \mathbb{N} : \right. \\ & \quad \left. u_n(x) - u(x) < \epsilon, n \geq N_\epsilon \right) \end{aligned}$$

The corresponding statement for increasing sequences is also true.

Proof. Take $\epsilon > 0$ arbitrary but fixed. We start with the set

$$C = \left\{ x \in \Omega \left| \begin{array}{l} \forall n \in \mathbb{N} : \\ u_n, u \text{ continuous at } x \end{array} \right. \right\}$$

which must have complement a set of First Baire Category, and hence it is dense. In view of (12), the set of points

$$C_\epsilon = \left\{ x \in C \left| \begin{array}{l} \exists N_\epsilon \in \mathbb{N} : \\ u_n(x) - u(x) < \epsilon, n \geq N_\epsilon \end{array} \right. \right\}$$

must be dense in Ω . From the continuity of u and the u_n on C it follows that

$$\begin{aligned} & \forall x_0 \in C_\epsilon : \\ & \exists \delta_{x_0} > 0 : \\ & x \in C, \|x - x_0\| < \delta \Rightarrow x \in C_\epsilon \end{aligned}$$

Since C is dense in Ω , the result follows. ■

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